Poisson's Equation in Electrostatics

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2007/2/4, 2010, 2011

Abstract

Poisson's equation is derived from Coulomb's law and Gauss's theorem. In mathematics, Poisson's equation is a partial differential equation with broad utility in electrostatics, mechanical engineering, and theoretical physics. It is named after the French mathematician, geometer and physicist Siméon-Denis Poisson (June 21, 1781 – April 25, 1840). Charles Augustin Coulomb (1736 -1806) was a French physicist who discovered an inverse relationship on the force between charges and the square of its distance. Karl Friedrich Gauss (1777-1855) was a German mathematician who is sometimes called the "prince of mathematics." He proved the fundamental theorem of algebra and the fundamental theorem of arithmetic.

1 Coulomb's Law, Electric Field, and Electric Potential

Electrostatics is the branch of physics that deals with the forces exerted by a static (i.e. unchanging) electric field upon charged objects [1]. The basic electrical quantity is charge ($e = -1.602 \times 10^{-19}$ $[C]$ electron charge in coulomb C). In a medium, an isolated charge $Q > 0$ located at $\mathbf{r}_0 = (x_0, y_0, z_0)$ produces an electric field $\mathbf{E}$ that exerts a force on all other charges. Thus, a charge $q > 0$ located at a different point $\mathbf{r} = (x, y, z)$ experiences a force from Q given by **Coulomb's law** [2] as

$$\mathbf{F} = q\mathbf{E} = q\frac{Q}{4\pi\varepsilon r^2}\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|} \qquad [N] , \tag{1.1}$$

where

$$
\begin{aligned}
&[N] \text{ represents the unit of the force in newton,}\\
r =& \ |\mathbf{r} - \mathbf{r}_0|\ , \text{ and}\\
\varepsilon =& \text{ permittivity (or dielectric constant) } [F/L] \text{ of the medium,}\\
F :& \ \text{farads } [F = C/V]\,.
\end{aligned}
$$

A *dimension* defines some physical characteristics. For example, length $[L]$, mass $[M]$, time $[T]$, velocity $[L/T]$, and force $[N = ML/T^2]$. A *unit* is a standard or reference by which a dimension can be expressed numerically. In SI (the International System of Units or in the French name Systeme Internationale d'Unites), the *meter* $[m]$, *kilogram* $[kg]$, *second* $[s]$, *ampere* $[A]$, *kelvin* $[K]$, and *candela* $[cd]$ are the base units for the six fundamental dimensions of length, mass, time, electric current, temperature, and luminous intensity [3].

The electric permittivity ε is conventionally expressed as

$$
\begin{aligned}
\varepsilon =& \ \varepsilon_r \varepsilon_0,\\
\varepsilon_r =& \text{ the dielectric constant or relative electric permittivity,}\\
\varepsilon_0 =& \text{ permittivity of vacuum} = 8.85 \times 10^{-12} \ [F/L]\,.
\end{aligned}
$$

For air at atmospheric pressure $\varepsilon_r = \varepsilon_{Air} = 1.0006$. Other dielectric constants $\varepsilon_{Water} = 80$, $\varepsilon_{\mathrm{Si}} = 11.68$, $\varepsilon_{InAs} = 14.55$, $\varepsilon_{GaAs} = 13.13$.

The **electric field** $\mathbf{E}(\mathbf{r})$ (a **force per unit charge**) produced by Q at $\mathbf{r}_0$ and felt by the unit charge at $\mathbf{r}$ is thus defined by

$$
\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\varepsilon r^2} \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|} \qquad [N/C]\,. \tag{1.2}
$$

By moving a charge q against the field between the two points a and b with a distance Δx, work is done. That is, for 1D case,

$$
\begin{aligned}
F\Delta x =& \ qE\Delta x\\
\Delta\Phi \ :=& \ E\Delta x = \frac{\text{force}}{\text{charge}} \times \text{distance} = \frac{\text{work}}{\text{charge}}\\
&\left[V = \frac{J}{C}, \ V = \text{volt}, \ J = NL \text{ joule}\right].
\end{aligned} \tag{1.3}
$$

This work per charge defines the **electric potential difference** $\Delta\Phi$ between the points a and b. Here the Greek letter delta Δ means a difference. This symbol will also be used in another meaning for the Laplace operator below.

Potential energy exists whenever an object which has charge (or mass) and has a position within an electric (or gravitational) field. Equivalently, the electric field can be defined by the potential as

$$
\begin{aligned}
\mathbf{E}(x) &= -\frac{\Delta\Phi}{\Delta x} = -\lim_{\Delta x \to 0} \frac{\Phi(x+\Delta x)-\Phi(x)}{\Delta x} = -\frac{d\Phi(x)}{dx} \quad [\text{in } 1D] \\
\mathbf{E}(\mathbf{r}) &= -\text{ grad } \Phi(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) \quad [\text{in } 2D \text{ or } 3D] \\
&= -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\Phi(\mathbf{r}).
\end{aligned} \tag{1.4}
$$

The negative sign above reminds us that moving against the electric field results in positive work. The **electric potential** $\Phi(\mathbf{r})$ is therefore a scalar function of $\mathbf{r}$ and is the (positive) work per unit charge that we must pay for moving the unit charge from infinity to the position $\mathbf{r}$ within the electric field generated by Q.

2 Gauss's Theorem

Gauss's Theorem is a 3D generalization from the Fundamental Theorem of Calculus in 1D. The following theorems can be found in standard Calculus books.

Theorem 1 (Fundamental Theorem of Calculus) *If f is a differentiable function on $[a, b]$, then*

$$
\begin{aligned}
&\int_a^b f'(x)dx = f(b) - f(a). \\
&\textit{Line Integral (1D)} = \textit{Point Evaluation (0D)}
\end{aligned} \tag{2.1}
$$

Theorem 2 (Divergence Theorem of Gauss (1832)) *Let B be an open bounded domain in $\mathbf{R}^3$ with a piecewise smooth boundary ∂B. Let $\mathbf{u}(x, y, z)$ be a differentiable vector function in B. Then*

$$
\begin{aligned}
\iiint_B div\ \mathbf{u}d\mathbf{r} = \iiint_B \nabla \cdot \mathbf{u}d\mathbf{r} &= \iint_{\partial B} \mathbf{u}\cdot\mathbf{n}dS \\
\textit{Domain Integral (3D)} &= \textit{Boundary Integral (2D)} \\
\textit{Total Mass Change in } B &= \textit{Mass Flows across } \partial B,
\end{aligned} \tag{2.2}
$$

where $\mathbf{n}$ is an outward unit normal vector on ∂B. The integral $\iint_{\partial B} \mathbf{u}\cdot\mathbf{n}dS$ is also called the **flux u** *across the surface S.*

Theorem 3 (Mean Value Theorem (MVT)) *If f is continuous on $[a,b]$, then there exists a number $c \in [a,b]$ such that*

$$\int_a^b f(x)dx = f(c)\int_a^b dx = f(c)\,(b-a)\,. \tag{2.3}$$

3 Poisson's Equation

Assume that the electric field $\mathbf{E}(\mathbf{r})$ is differentiable in its domain $\Omega \subset \mathbf{R}^3$. To simplify our presentation of using Gauss's theorem, we consider any subset $B \subset \Omega$ as a ball with radius r centered at $\mathbf{r}_0$, i.e., $B = \{\mathbf{r} \in \Omega : |\mathbf{r} - \mathbf{r}_0| < r\}$ whose boundary ∂B is a sphere. The following argument can be generalized to any closed bounded domain as required by Causs's theorem with, of course, more technicalities [2]. We may think for the moment that the variable $\mathbf{r}_0$ is "fixed." Applying (2.2) to the electric field in (1.4), we have

$$\begin{aligned}
\lim_{r\to 0}\iiint_B \nabla\cdot\varepsilon\mathbf{E}(\mathbf{r})d\mathbf{r} &= -\lim_{r\to 0}\iiint_B \nabla\cdot\varepsilon\nabla\Phi(\mathbf{r})d\mathbf{r} && (3.1)\\
&= -\nabla\cdot\varepsilon\nabla\Phi(\mathbf{r}_0)\left[\lim_{r\to 0}\iiint_B d\mathbf{r}\right] \text{ by MVT} && (3.2)\\
&= -\nabla\cdot\varepsilon\nabla\Phi(\mathbf{r}_0)\left[\lim_{r\to 0}\frac{4\pi r^3}{3}\right] && (3.3)\\
&= \lim_{r\to 0}\iint_{\partial B}\varepsilon\mathbf{E}(\mathbf{r})\cdot\mathbf{n}dS. && (3.4)
\end{aligned}$$

Eq. (3.2) is valid if $\nabla\cdot\varepsilon\nabla\Phi(\mathbf{r})$ is continuous on B. Since B is infinitesimally small, the existing point guaranteed by MVT is nothing but $\mathbf{r}_0$ as $r \to 0$.

The surface integral

$$\psi = \iint_{\partial B}\varepsilon\mathbf{E}\cdot\mathbf{n}dS \quad [C] \tag{3.5}$$

is called the **electric flux** of $\mathbf{E}$ through the sphere ∂B. We can imagine the flux ψ as the sum (total charge) of infinitely many small charges that have been "flowing" through infinitesimal surfaces dS (note the unit in Coulomb). Note also that (1.2) gives us a hint that

$$\psi = 4\pi r^2 \times \varepsilon E = Q \tag{3.6}$$

is the total charge Q passing through the sphere ∂B. There are several situations we need to be more specific about the limiting flux in (3.4).

Case 1. *No charges within B and Ω at all.*

Nothing is passing through the sphere ∂B. Before taking the limit in both (3.3) and (3.4), we have zero for (3.4) which implies that

$$-\nabla \cdot \varepsilon \nabla \Phi(\mathbf{r}_0) = 0 \text{ for all } \mathbf{r}_0 \in \Omega \tag{3.7}$$

provided that any ball containing $\mathbf{r}_0$ does not have any charge in it. Let us assume for the moment that the domain Ω contains nothing and the permittivity is constant. We then have the **Laplace equation**

$$\Delta \Phi(\mathbf{r}) = 0 \text{ for all } \mathbf{r} \in \Omega, \tag{3.8}$$

where $\Delta = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator. One may wonder that the solution of (3.8) is identically zero. Yes or no, it depends on the boundary condition of (3.8). Note carefully that we apply Gauss's theorem for an *open* domain Ω and say nothing about the charges on the boundary $\partial\Omega$. This is one of examples that we must be very rigorous on the precise statement of mathematical definitions and theorems. In general, the **Laplace problem** can be stated as to find a solution (potential) Φ such that

$$\Delta \Phi(\mathbf{r}) = 0, \qquad \forall\, \mathbf{r} \in \Omega, \tag{3.9a}$$

$$\Phi(\mathbf{r}) = \Phi_D(\mathbf{r}), \qquad \forall\, \mathbf{r} \in \partial\Omega_D, \tag{3.9b}$$

$$\frac{\partial \Phi(\mathbf{r})}{\partial n} = \Phi_N(\mathbf{r}), \qquad \forall\, \mathbf{r} \in \partial\Omega_N, \tag{3.9c}$$

where $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, Φ_D and Φ_N are given functions. (3.9b) is usually called a *Dirichlet* (or essential) boundary condition whereas (3.9c) called a *Neumann* (or natural) boundary condition. Here $\frac{\partial \Phi(\mathbf{r})}{\partial n} = \nabla \Phi(\mathbf{r}) \cdot \mathbf{n}$ is a directional derivative with the direction of the outward normal unit $\mathbf{n}$ to $\partial\Omega_N$ at $\mathbf{r}$.

Case 2. *B contains a single charge Q_0 at $\mathbf{r}_0$ and Ω contains finitely many charges Q_i at $\mathbf{r}_i$, $i = 1, \cdots n$.*

From (3.3) and (3.4), we have

$$-\nabla \cdot \varepsilon \nabla \Phi(\mathbf{r}_0) \left[\lim_{r \to 0} \frac{4\pi r^3}{3} \right] = \lim_{r \to 0} Q_0 = Q_0$$

which yields a very strange equation

$$-\nabla \cdot \varepsilon \nabla \Phi(\mathbf{r}_0) = \frac{Q_0}{\lim_{r \to 0} \frac{4\pi r^3}{3}} \tag{3.10}$$

The term $\frac{Q_0}{\lim_{r \to 0} \frac{4\pi r^3}{3}}$ has no physical meaning and can only be *abstracted* mathematically by means of the Dirac delta function [1] (a distribution as called by mathematicians)

$$\delta(\mathbf{r}-\mathbf{r}_0)=\begin{cases}\infty, \ \mathbf{r}=\mathbf{r}_0\\ 0, \ \mathbf{r}\neq\mathbf{r}_0\end{cases}; \qquad \int_{\mathbf{R}^3}\delta(\mathbf{r}-\mathbf{r}_0)d\mathbf{r}=1. \tag{3.11}$$

For any continuous function $Q(\mathbf{r})$, the delta function has the fundamental property that

$$\int_{\mathbf{R}^3} Q(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}_0)d\mathbf{r}=Q(\mathbf{r}_0) \tag{3.12}$$

Since the volume $\lim_{r\to 0}\frac{4\pi r^3}{3}$ is infinitely small, we can write $\lim_{r\to 0}\frac{4\pi r^3}{3}=d\mathbf{r}$ and define $\rho(\mathbf{r})$ as a continuous function such that

$$\rho(\mathbf{r})=\frac{Q_0}{d\mathbf{r}} \quad \left[CL^{-3}\right] \tag{3.13}$$

This function is thus called a *charge density function.* This is not a mathematically perfect way to define the density function for a point charge because the volume $d\mathbf{r}$ is not definite. A more general definition for all kinds of charges including the point charge is the following. We define the density function for a point charge Q_0 at $\mathbf{r}_0$ by means of delta function as

$$\rho(\mathbf{r})=Q_0\delta(\mathbf{r}-\mathbf{r}_0) \tag{3.14}$$

By (3.12), we thus have

$$\int_{\mathbf{R}^3}\rho(\mathbf{r})d\mathbf{r}=\int_{\mathbf{R}^3}Q_0\delta(\mathbf{r}-\mathbf{r}_0)d\mathbf{r}=Q_0 \tag{3.15}$$

Consequently, we have the following **Poisson equation** for a point charge

$$-\nabla\cdot\varepsilon\nabla\Phi(\mathbf{r})=Q_0\delta(\mathbf{r}-\mathbf{r}_0) \tag{3.16}$$

It should be noticed that the delta function in this equation implicitly defines the density which is important to correctly interpret the equation in actual physical quantities.

If the domain Ω contains isolated charges Q_i at $\mathbf{r}_i$, $i=1,2,\cdots,n$, the **Poisson equation** becomes

$$-\nabla\cdot\varepsilon\nabla\Phi(\mathbf{r})=\sum_{i=1}^{n}Q_i\delta(\mathbf{r}-\mathbf{r}_i) \tag{3.17}$$

Case 3. *Both B and Ω contain infinitely many charges expressed by a density function $\rho(\mathbf{r})$.*

It is clear now that the total flux charges of (3.5) "flowing" through the sphere ∂B is equal to the total charges residing in the open ball B, i.e., by (3.1), (3.4), and (3.5)

$$\psi = \iint_{\partial B} \varepsilon \mathbf{E} \cdot \mathbf{n} dS = \iiint_B \rho(\mathbf{r}) d\mathbf{r} = - \iiint_B \nabla \cdot \varepsilon \nabla \Phi(\mathbf{r}) d\mathbf{r} \tag{3.18}$$

These equalities hold for any arbitrary domain B. The last equality therefore implies the **Poisson problem**

$$-\nabla \cdot \varepsilon \nabla \Phi(\mathbf{r}) = \rho(\mathbf{r}), \qquad \forall\, \mathbf{r} \in \Omega, \tag{3.19a}$$

$$\Phi(\mathbf{r}) = \Phi_D(\mathbf{r}), \qquad \forall\, \mathbf{r} \in \partial\Omega_D, \tag{3.19b}$$

$$\frac{\varepsilon \partial \Phi(\mathbf{r})}{\partial n} = \Phi_N(\mathbf{r}), \qquad \forall\, \mathbf{r} \in \partial\Omega_N. \tag{3.19c}$$

References

[1] Wikipedia: The Free Encyclopedia, http://en.wikipedia.org/wiki/Electrostatics

[2] J. D. Jackson, Classical Electrodynamics, 3rd ed., Wiley, 1999.

[3] J. D. Kraus and D. A. Fleisch, Electromagnetics with Applications, 5th ed., McGraw-Hill, 1999.